On the non-linear energy transfer in a gravity wave spectrum

Part 2. Conservation theorems; wave-particle analogy; irreversibility

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(Received 7 August 1962)

From the conditions of energy and momentum conservation it is shown that all four interaction coefficients of the elementary quadruple interactions discussed in Part I of this paper are equal. A further conservative quantity is then found which can be interpreted as the number density of the gravity-wave ensemble representing the random sea surface. The well-known analogy between a random linear wave field and a mass particle ensemble is found to hold also in the case of weak non-linear interactions in the wave field. The interaction conditions for energy transfer between waves to correspond to the equations of energy and momentum conservation for a particle collision, the final equation for the rate of change of the wave spectrum corresponding to Boltzmann's equation for the rate of change of the number density of an ensemble of colliding mass particles. The stationary wave spectrum corresponding to the Maxwell distribution for a mass particle ensemble is found to be degenerate. The question of the irreversibility of the transfer process for gravity waves is discussed but not completely resolved.

1. Conservation theorems

In Hasselmann (1962; referred to in the following as I) it was shown that the non-linear coupling between the spectral components of a random, homogeneous sea gives rise to an energy transfer

$$\frac{\partial F_4}{\partial t} = \int \cdots \int_{-\infty}^{+\infty} \frac{9\pi g^2 . D_4 \omega_4}{4\rho^2(\omega_1\omega_2\omega_3\omega_4)^2} \{D_4\omega_4 F_1 F_2 F_3 + D_3\omega_3 F_1 F_2 F_4 - D_2\omega_2 F_1 F_3 F_4 - D_1\omega_1 F_2 F_3 F_4\}\, \delta(\omega_1+\omega_2-\omega_3-\omega_4)\, \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\, d^2k_1 d^2k_2 d^2k_3, \tag{1.1}$$

where $F_j = F(\mathbf{k}_j)$ is the energy spectrum in terms of wave-number, defined such that only waves travelling in the positive $\mathbf{k}_j$-direction contribute to the spectrum at $\mathbf{k}_j$. The energy transfer results from resonant coupling between four wave components, the δ-functions expressing the fact that resonance occurs only if the components satisfy the conditions

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4, \tag{1.2}$$

$$\omega_1 + \omega_2 = \omega_3 + \omega_4, \tag{1.3}$$

where $\omega_j = (gk_j \tanh k_j h)^{\frac{1}{2}}$ is the free-wave frequency (h = water depth). Equation (1.1) is equivalent to equation (4.29) of I, the integration variables having been suitably interchanged in order to obtain common δ-functions and the interaction coefficients renamed as follows:

$$D_1 = D^{+\,+\,-}_{\mathbf{k}_3,\mathbf{k}_4,-\mathbf{k}_2}, \quad D_2 = D^{+\,+\,-}_{\mathbf{k}_3,\mathbf{k}_4,-\mathbf{k}_1}, \quad D_3 = D^{+\,+\,-}_{\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_4}, \quad D_4 = D^{+\,+\,-}_{\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_3},$$

where $D^{+\,+\,-}_{\mathbf{k}_a,\mathbf{k}_b,-\mathbf{k}_c}$ is the interaction coefficient (I, 4.9) determining the rate of energy transfer from three interacting components $\mathbf{k}_a$, $\mathbf{k}_b$, $\mathbf{k}_c$ to a fourth component $\mathbf{k}_a + \mathbf{k}_b - \mathbf{k}_c$.

Since the transfer process conserves both energy and momentum, equation (1.1) must satisfy the conditions

$$\iint_{-\infty}^{+\infty} \frac{\partial F_4}{\partial t} d^2k_4 = 0, \tag{1.4}$$

$$\iint_{-\infty}^{+\infty} \frac{\partial F_4}{\partial t} \frac{\mathbf{k}_4}{\omega_4} d^2k_4 = 0. \tag{1.5}$$

Equation (1.5) requires some justification, since although it was shown in I that the energy redistribution resulting from non-stationary higher-order perturbations could be interpreted up to the sixth order simply as a change in the spectrum of the linear wave field, it is not immediately apparent whether this holds also for the momentum. The mean momentum per unit projection area is

$$\overline{\rho \int_{-h}^{\zeta} \mathbf{v}\, dz} = \rho \int_{-h}^{0} \bar{\mathbf{v}}\, dz + \rho \overline{\int_{0}^{\zeta} \mathbf{v}\, dz} \tag{1.6}$$

where $\mathbf{v}$ is the velocity, ρ the density and ζ the surface displacement. To the lowest order, the first term on the right-hand side of (1.6) is constant and can be made zero by suitable choice of co-ordinate systems, whereas the second term is $\rho\overline{\mathbf{v}\zeta}$, yielding, the well-known expression

$$\iint_{-\infty}^{+\infty} F(\mathbf{k})\,(\mathbf{k}/\omega)\, d^2k$$

for the momentum of the linearized wave field. Since the second term is quadratic to the lowest order it is readily verified that the analysis of the higher-order perturbations of this term follows along exactly the same lines as that in I for the energy, which is also quadratic to the lowest order. The result is thus expressible in terms of the spectrum of the linear solution. It remains to be shown that the first term in (1.6) has no non-stationary components up to the sixth order. This follows immediately from the general form of the perturbation equations (I, 1.23) for the velocity potential. An nth order velocity potential of the form $a(t)\,x + b(t)\,y$ as required for $\bar{\mathbf{v}} \neq 0$ cannot be generated by interactions between lower-order solutions, as these produce only harmonic components, provided they themselves are harmonic. Thus perturbation solutions with $\bar{\mathbf{v}} \neq 0$ can enter only as solutions of the homogeneous perturbation equations via the higher-order initial conditions, and can be made zero by suitable choice of the co-ordinate system. (This was, in fact, implicitly assumed in I by stating the initial and boundary conditions only at the free surface and not at infinity.)

If (1.1) is substituted in (1.4) and (1.5) and the transformations

$$\begin{array}{lll} \mathbf{k}_1 \to \mathbf{k}_1, & \mathbf{k}_1 \to \mathbf{k}_3, & \mathbf{k}_1 \to \mathbf{k}_4 \\ \mathbf{k}_2 \to \mathbf{k}_2, & \mathbf{k}_2 \to \mathbf{k}_4, & \mathbf{k}_2 \to \mathbf{k}_3 \\ \mathbf{k}_3 \to \mathbf{k}_4, & \mathbf{k}_3 \to \mathbf{k}_1, & \mathbf{k}_3 \to \mathbf{k}_1 \\ \mathbf{k}_4 \to \mathbf{k}_3, & \mathbf{k}_4 \to \mathbf{k}_2, & \mathbf{k}_4 \to \mathbf{k}_2 \end{array}$$

are carried out in turn for each equation, the sums of the resultant and original expressions yield the relations (the first of which was mentioned also in I)

$$D_1 \omega_1 + D_2 \omega_2 = D_3 \omega_3 + D_4 \omega_4, \tag{1.7}$$

$$D_1 \mathbf{k}_1 + D_2 \mathbf{k}_2 = D_3 \mathbf{k}_3 + D_4 \mathbf{k}_4. \tag{1.8}$$

Equations (1.7) and (1.8) represent a system of three homogeneous equations for four unknowns and therefore uniquely determine the ratios of the interaction coefficients D_j, since it is easily verified that the rank of the system is three except for isolated points which may be ignored for continuous D_j. By comparison of (1.2) and (1.3) with (1.7) and (1.8) the solution is immediately seen to be

$$D_1 = D_2 = D_3 = D_4 = D. \tag{1.9}$$

This rather surprising result was hardly to have been anticipated from the lengthy algebraic expression (I, 1.49, 1.50) for the interaction coefficients.

Apart from simplifying (1.1), the relations (1.9) lead to a very simple picture for the net energy and momentum balance between four interacting wave trains. The resultant energy flux is always such that the components on one side of the interaction equations (1.2) and (1.3) either both lose or both gain energy, the rate of loss or gain of a particular component being proportional to its frequency. The same is true for the net momentum balance, the transfer rates in this case being proportional to the wave-numbers. At a first glance it would seem that the relations can be applied further to achieve some simplification of the analysis in I. It was possible there to evaluate the energy gain of a given component as a result of the direct interaction between three other components from a perturbation analysis of only the third order, whereas in order to evaluate the energy losses of each of the three interacting components the analysis had to be extended further to the fifth order. From the above it now appears that it is sufficient to evaluate only the energy gain of the first component, the energy losses of the remaining three components then being determined from considerations of energy and momentum conservation. (The conservation arguments can be formulated independent of the final form (1.1) of the energy transfer for an elementary group of four interacting wave trains.) However, this argument cannot be carried through rigorously, for the third-order perturbation analysis yielded not only an energy perturbation term increasing linearly with t, as to be expected for a transfer expression of the form (1.1), but also a spurious term increasing proportional to t^2. This would normally dominate over the linear term, but was found to cancel against a similar term when the analysis was carried through completely to the fifth order.

Equation (1.9) is all that can be inferred from the conservation of mass, momentum and energy, as the additional condition of mass conservation requires simply that the mean surface elevation remains constant, and this can readily be seen to be satisfied. However, it can be shown that a further conservative quantity exists, the mean 'number density'

$$\bar{n} = \iint_{-\infty}^{+\infty} n(\mathbf{k})\, d^2k \tag{1.10}$$

where $$n(\mathbf{k}) = F(\mathbf{k})/\alpha\omega, \tag{1.11}$$

with α an arbitrary constant. The equation for the rate of change of $n(\mathbf{k})$ has the symmetrical form

$$\frac{\partial n_4}{\partial t} = \int \dots \int a\{n_1 n_2(n_3+n_4) - n_3 n_4(n_1+n_2)\}\, d^2k_1\, d^2k_2\, d^2k_3, \tag{1.12}$$

where $$a = \frac{9\pi g^2 D^2 \alpha^2}{4\rho^2 \omega_1 \omega_2 \omega_3 \omega_4}\, \delta(\omega_1+\omega_2-\omega_3-\omega_4)\, \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4). \tag{1.13}$$

From (1.12) and (1.13) it is immediately seen that

$$\iint_{-\infty}^{+\infty} \frac{\partial n_4}{\partial t}\, d^2k_4 = 0. \tag{1.14}$$

Equation (1.12) has formal similarity with the Boltzmann equation for the rate of change of the number density, in phase space, of a spacially homogeneous distribution of mass particles, except that the collision probability is proportional to a cubic rather than a quadratic expression in the 'number densities' $n(\mathbf{k})$. It will be seen in the next section that this analogy can be extended and is related to the equivalence of wave and ray (or particle) representations of wave fields in which the mean quantities vary slowly in comparison to the flucutating quantities.

2. Wave-particle analogy

Instead of representing a homogeneous, Gaussian sea as a superposition of an infinite number of statistically independent, infinitely long wave trains we may also, to any desired degree of approximation, consider it to be a superposition of a very large number of statistically independent wave *groups*, the dimension of the wave groups being large in comparison to their wavelength. Each wave group then propagates with its appropriate group velocity, preserving its shape over a distance large in comparison to the dimension of the group. It is well known that in the general case in which the phase velocity of the wave-propagating medium varies with position, the propagation laws of wave groups can be expressed in a form completely analogous to the Hamiltonian equations of motion for a mass particle in a force field, the frequency and wave-number of a wave group corresponding to the energy and momentum of a particle. This suggests (although the choice is otherwise arbitrary) that we define the energy E of a wave group of our ensemble as $E = \alpha\omega$, where α is a (very small) constant. Since the ratio of the momentum to the energy of a wave group is equal to the phase velocity, the momentum of the wave group is $\mathbf{p} = \alpha\mathbf{k}$, in accordance with the wave-particle analogy. The number density of the wave group ensemble in $(\mathbf{k}, \mathbf{x})$ phase space is then $n(\mathbf{k}) = F(\mathbf{k})/\alpha\omega$. In our case of a homogeneous wave field, $n(\mathbf{k})$ is independent of $\mathbf{x}$, but the number density can clearly be defined in the same way for the more general case of a quasi-homogeneous wave field in a fluid of finite, slowly varying depth, and it is then seen that there is a complete analogy between an ensemble of non-interacting wave groups and an ensemble of non-colliding particles in a force field.†

It is interesting now to note that this analogy appears to hold also in the case of non-linear coupling between the wave components, the interactions between waves corresponding to particle collisions. The interaction conditions (1.2) and (1.3) may be interpreted as the equations for the conservation of momentum and energy of two colliding particles, the index pairs 1, 2 and 3, 4 corresponding to the pair of particles before and after the collision respectively. A two-particle collision of this form also automatically conserves the mean number density $\bar{n}$ in the $\mathbf{x}$-plane, as required by (1.14). The interpretation of the wave interaction conditions as the momentum and energy conservation laws of particle collisions probably generalizes to arbitrary wave fields with weak non-linearities. However, the number of particles before and after a collision will generally not be the same (for example, they are not the same if the energy transfer is of the second rather than the third order), so that the conservation of the mean number density in our case should be regarded as a coincidence.

Up to this point there is a complete analogy between an ensemble of interacting wave groups and an ensemble of colliding elastic particles. However, on comparing equation (1.12) with the corresponding Boltzmann equation for a mass particle ensemble we find a basic difference: the probability of two wave groups 'colliding' is proportional not only to $n(\mathbf{k}_1)$ and $n(\mathbf{k}_2)$, as in the case of a mass particle ensemble, but also to $n(\mathbf{k}_3)+n(\mathbf{k}_4)$, the probability of the inverse collision taking place being proportional to $n(\mathbf{k}_3)\,n(\mathbf{k}_4)\{n(\mathbf{k}_1)+n(\mathbf{k}_2)\}$, with the same factor of proportionality (in analogy again to the corresponding symmetry of the Boltzmann equation). We may interpret this as an interaction between two colliding wave groups $\mathbf{k}_1$ and $\mathbf{k}_2$ taking place only if a wave group $\mathbf{k}_3$ or $\mathbf{k}_4$ is present to act as a 'catalyst'.

The difference between the collision probabilities for mass-particle and gravity-wave ensembles has important consequences for the development of the distribution functions. For every continuous initial distribution, the asymptotic solution of the Boltzmann equation is the stationary Maxwell distribution. The corresponding stationary solution in the case of a gravity-wave ensemble will be found to be singular, and the question as to whether a given initial distribution approaches this solution asymptotically is not as simple to answer.

† For example, it has been pointed out by G. Backus (private communication) that Longuet-Higgins's (1957) result that the spectral density remains invariant under wave-number transformations due to refraction follows directly from this analogy and Liouville's theorem for a particle ensemble.

3. Stationary solutions, irreversibility

A stationary solution of equation (1.12) can be obtained by letting the integrand on the right-hand side vanish identically, the distribution satisfying the equation

$$\frac{n(\mathbf{k}_1)\,n(\mathbf{k}_2)}{n(\mathbf{k}_1)+n(\mathbf{k}_2)} = \frac{n(\mathbf{k}_3)\,n(\mathbf{k}_4)}{n(\mathbf{k}_3)+n(\mathbf{k}_4)}, \tag{3.1}$$

for all $\mathbf{k}_j$ satisfying the interaction conditions (1.2), (1.3). This is possible only if

$$\frac{n(\mathbf{k}_1)\,n(\mathbf{k}_2)}{n(\mathbf{k}_1)+n(\mathbf{k}_2)} = f(\omega_1+\omega_2, \mathbf{k}_1+\mathbf{k}_2) = f(\omega_3+\omega_4, \mathbf{k}_3+\mathbf{k}_4) = \frac{n(\mathbf{k}_3)\,n(\mathbf{k}_4)}{n(\mathbf{k}_3)+n(\mathbf{k}_4)}, \tag{3.2}$$

with a suitable function f.

Solving for $n(\mathbf{k}_2)$, we have

$$n(\mathbf{k}_2) = \frac{n(\mathbf{k}_1)f}{n(\mathbf{k}_1)-f},$$

and after differentiation with respect to k_{1_i},

$$\frac{\partial}{\partial\tilde{\omega}}\left(\frac{1}{f}\right)c_{g_i}(\mathbf{k}_1) + \frac{\partial}{\partial\tilde{k}_i}\frac{1}{f} = \frac{\partial}{\partial k_{1_i}}\left(\frac{1}{n(\mathbf{k}_1)}\right), \tag{3.3}$$

where $\tilde{\omega} = \omega_1+\omega_2$, $\tilde{\mathbf{k}} = \mathbf{k}_1+\mathbf{k}_2$ and $\mathbf{c}_g$ is the group velocity $(\mathbf{k}/k)\,d\omega/dk$. Differentiating now with respect to k_{2_j}, we obtain

$$\frac{\partial^2}{\partial\tilde{\omega}^2}\left(\frac{1}{f}\right)c_{g_i}(\mathbf{k}_1)\,c_{g_j}(\mathbf{k}_2) + \frac{\partial^2}{\partial\tilde{\omega}\partial\tilde{k}_i}\left(\frac{1}{f}\right)c_{g_j}(\mathbf{k}_2) + \frac{\partial^2}{\partial\tilde{\omega}\,\partial\tilde{k}_j}\left(\frac{1}{f}\right)c_{g_i}(\mathbf{k}_1) + \frac{\partial^2}{\partial\tilde{k}_i\partial\tilde{k}_j}\left(\frac{1}{f}\right) = 0. \tag{3.4}$$

Since $\mathbf{k}_1$ and $\mathbf{k}_2$ may be varied while $\mathbf{k}_1+\mathbf{k}_2$ and $\omega_1+\omega_2$ are kept constant, and it can readily be seen that the corresponding variations in $c_{g_i}(\mathbf{k}_1)$ and $c_{g_j}(\mathbf{k}_2)$ are not such that they satisfy a quadratic equation of the form (3.4), equation (3.4) can be satisfied only if all the coefficients vanish,

$$\frac{\partial^2}{\partial\tilde{\omega}^2}\left(\frac{1}{f}\right) = \frac{\partial^2}{\partial\tilde{\omega}\,\partial\tilde{k}_i}\left(\frac{1}{f}\right) = \frac{\partial^2}{\partial\tilde{k}_i\partial\tilde{k}_j}\left(\frac{1}{f}\right) = 0. \tag{3.5}$$

The general solution of (3.5) is $f = (b'+c\tilde{\omega}+d_i\tilde{k}_i)^{-1}$, where b', c and d_i are constants. From (3.3), the general solution of $n(\mathbf{k})$ is then

$$n(\mathbf{k}) = (b+c\omega+d_i k_i)^{-1} \tag{3.6}$$

The moments of the solution (3.6) corresponding to the mean number, energy and momentum density do not exist. Moreover, for $d_i \neq 0$ the solution becomes negative in the region of the $\mathbf{k}$-plane to one side of the curve $b+c\omega+d_i k_i = 0$, and is singular on the curve itself. Thus (3.6) cannot be the asymptotic solution of an initial distribution corresponding to a finite mean energy and momentum unless we let the constants in (3.6) tend to infinity in such a manner that the mean number, energy and momentum density become finite, whilst n remains positive for finite $\mathbf{k}$. This would require $n(\mathbf{k}) \to 0$ for fixed $\mathbf{k}$.

The question then arises as to whether the solutions (3.6) are the only stationary solutions of (1.12) In the case of the Boltzmann equation, the corresponding solution obtained by setting the integrand of the collision integral identically equal to zero is the Maxwell distribution, and the H-theorem establishes that this is indeed the only continuous stationary solution. A theorem analogous to the H-theorem can be derived for our case also, but the situation is no longer quite so simple, since the expression corresponding to the entropy turns out to be a divergent integral. The argument of the irreversibility of the energy transfer and the uniqueness of the stationary solution can still be carried through in a somewhat limited form, however, if the 'entropy' is defined first as an integral over only a finite region of the phase space and the region is then allowed to increase indefinitely.

The usual proof of the H-theorem for the Boltzmann equation is based on finding an expression $H = -\int G(n)\,d\mathbf{c}$, where $\mathbf{c}$ is the particle phase space, such that $dH/dt = -\int G'(n)\,(\partial n/\partial t)\,d\mathbf{c}$ can be expressed as the integral of a positive definite expression in n when the Boltzmann collision formula is substituted for $\partial n/\partial t$, the expression vanishing only if the integrand of the collision integral vanishes identically. Considering then the corresponding expression

$$\frac{d\hat{H}}{dt} = -\iint_{-\infty}^{+\infty} G'(n)\,(\partial n/\partial t)\,d^2k, \tag{3.7}$$

in our case, we obtain after substitution of (1.1) for $\partial n/\partial t$ and symmetrizing in the same manner as in the derivation of (1.7) and (1.8)

$$\begin{aligned}\frac{d\hat{H}}{dt} = -\int\cdots\int \frac{a}{4}\left(\frac{n_1n_2}{n_1+n_2}-\frac{n_3n_4}{n_3+n_4}\right)(n_1+n_2)\,(n_3+n_4)\\ \times\{-G'(n_1)-G'(n_2)+G'(n_3)+G'(n_4)\}\,d^2k_1\,d^2k_2\,d^2k_3\,d^2k_4.\end{aligned} \tag{3.8}$$

If the integrand in (3.8) is to be positive for

$$\frac{n_1n_2}{n_1+n_2} \neq \frac{n_3n_4}{n_3+n_4},$$

the function G' must satisfy the inequalities

$$G'(n_1)+G'(n_2) \gtrless G'(n_3)+G'(n_4),$$

according as

$$\frac{n_1n_2}{n_1+n_2} \lessgtr \frac{n_3n_4}{n_3+n_4}.$$

This condition, in turn, can hold only if

$$G'(n_1)+G'(n_2) = J\left(\frac{n_1n_2}{n_1+n_2}\right) \tag{3.9}$$

where J is a monotonically strictly decreasing function. Differentiating (3.9) with respect to n_1, we obtain

$$G''(n_1)\,n_1^2 = \left(\frac{n_1n_2}{n_1+n_2}\right)^2 J'\left(\frac{n_1n_2}{n_1+n_2}\right). \tag{3.10}$$

Since the left- and right-hand sides of (3.10) are functions of different arguments, both must be constant. Hence G and J have the general form

$$G(n) = c_1 \ln n + c_2 n + c_3,$$

$$J(n) = \frac{c_1}{n} + 2c_2,$$

where the constants c_j have been adjusted to satisfy (3.9). Since J is required to decrease monotonically, c_1 must be positive. The constants c_2 and c_3 do not enter in (3.8) and may be taken as zero. Putting $c_1 = 1$, we thus have, formally,

$$\hat{H} = -\iint_{-\infty}^{+\infty} \ln n(k)\, d^2k,$$

where $d\hat{H}/dt < 0$ for solutions other than the stationary distribution (3.6). The latter can also be derived by formal variational analysis from the requirement that $\hat{H}$ is a minimum under the side conditions of constant mean number, energy and momentum density.

Unfortunately, $\hat{H}$ does not converge. The difficulty can be overcome to some extent by introducing the truncated integral

$$\hat{H}_R = -\iint_{|\mathbf{k}|<R} \ln n\; d^2k. \tag{3.11}$$

The time derivative of H_R can then be written in the form

$$\frac{\partial \hat{H}_R}{\partial t} = -\int \cdots \int_{|\mathbf{k}_4|<R} \frac{d^2k_4}{n_4}\Bigg\{\left(\int\cdots\int_{A_1} + \int\cdots\int_{A_2}\right) \times a[n_1 n_2(n_3+n_4) - n_3 n_4(n_1+n_2)]\, d^2k_1\, d^2k_2\, d^2k_3\Bigg\},$$

where A_1 is the region $|\mathbf{k}_1| < R$, $|\mathbf{k}_2| < R$, $|\mathbf{k}_3| < R$ and A_2 is the complementary region of ($\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$)-space. The product of the region $|\mathbf{k}_4| < R$ with A_1 is then symmetrical in all four wave-numbers, so that the integral can be symmetrized as above, giving

$$\begin{aligned}\frac{\partial \hat{H}_R}{\partial t} = \int \cdots \int_{|\mathbf{k}_i|<R} \tfrac{1}{4} a[n_1 n_2(n_3+n_4) - n_3 n_4(n_1+n_2)] \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4}\right) \\ \times d^2k_1\, d^2k_2\, d^2k_3\, d^2k_4 \\ - \int \cdots \int_{\substack{|\mathbf{k}_4|<R \\ A_2}} a[n_1 n_2(n_3+n_4) - n_3 n_4(n_1+n_2)]\, n_4^{-1}\, d^2k_1\, d^2k_2\, d^2k_3\, d^2k_4.\end{aligned}$$

By rather tedious expansion of the transfer coefficient a (part 3) it can be shown that for distributions which fall off asymptotically at a rate more rapidly than $k^{-4\cdot 75}$ the second integral becomes small in comparison to the first as $R \to \infty$. Since the first integral is negative except for distributions of the form (3.6) (which decay more slowly than $k^{-4\cdot 75}$) we thus have for sufficiently large R

$$\frac{d\hat{H}_R}{dt} < 0.$$

For H_R to be finite, we must exclude distributions which vanish in some regions of the $\mathbf{k}$-plane. However, if the area of the region in which $n \neq 0$ is non-zero it is readily seen that dn/dt must be positive in at least part of the region $n = 0$, for it is then always possible to find an interaction in which energy is transferred from three wave-numbers lying just outside the region $n = 0$ to a resultant wave-number just within. Thus these distributions are also non-stationary. If the area of the region in which $n \neq 0$ is zero, i.e. if n is a discrete superposition of δ-functions, this argument does not hold and one can then indeed find distributions which are stationary, for example, if the wave-numbers of the δ-functions are non-interacting or the energies of interacting components are equal. However, these are unstable limiting cases in the sense that a slight broadening of the spectral lines leads rapidly to a further line broadening accompanied with a decrease of the spectral peak.† They are thus analogous to the degenerate stationary solution of the Boltzmann equation in which all particle velocities are equal and are of no significance for the question of irreversibility.

Unfortunately, the restriction imposed on the asymptotic behaviour excludes distributions which may be feasible physically. For the mean square wave slope to be finite, for example, the exponent need only be smaller than $-4{\cdot}5$. It is not clear whether stationary solutions other than (3.6) exist for exponents greater than $-4{\cdot}75$. The question is of interest, in particular as regards the possible existence of stationary distributions corresponding to Kolmogoroff's inertial subrange for turbulent spectra. It may be of interest in this connexion that the condition $n(\mathbf{k}) > O(\mathbf{k}^{-4\cdot 75})$ is identical with the condition for a predominately local energy transfer at high wave-numbers (part 3). None the less, it is felt that solutions of this type probably do not exist, as it is difficult to see how a constant energy flux from an energy source at zero wave-number to a sink at infinity can conserve both the mean energy $\bar{E} = \alpha\iint n(\mathbf{k})\,\omega\,d^2k$ and the mean number density $\bar{n} = \iint n(\mathbf{k})\,d^2k$.

There is then some evidence that the only stationary solutions of (1.12) are the degenerate distributions (3.6), and that all other distributions tend to these irreversibly. It may appear surprising that despite the general analogy between a gravity-wave and mass-particle ensemble there should be such a marked difference in the form of the stationary distributions, since the Maxwell–Boltzmann distribution holds not only for a mass-particle ensemble but under very general conditions for any system which can be described in terms of a Hamiltonian. However, the basic difference can be attributed to exactly this fact. Although for a gravity-wave ensemble the evolution of a microsystem (i.e. the propagation of a wave group) can be represented in a Hamiltonian form, this is not generally the case for the macrosystem consisting of a number of interacting wave groups, since the energy transfer between a discrete set of wave groups depends not only on the momentum and position of the wave groups but also on their relative phases. The dependence on phase vanishes only in the case of a Gaussian ensemble.

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† This may be expected intuitively, but was also tested by computation.